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Characteristic functions of quantum graphs

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Abstract
In this paper, we give a reduction formula for the characteristic functions of the Sturm–Liouville boundary value problems defined on a tree. We also discuss the multiplicity of the eigenvalues and interlacing properties between two spectral sets associated with different boundary conditions.

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1. Introduction

The theory of quantum graphs, i.e. Schrödinger or Dirac operators defined on graph domains, is a fast developing research direction in the field of ordinary differential equations (see, e.g., [4, 5, 7, 12–16]). Similar problems also occur in the theory of small transversal vibrations of webs of strings or rods (see, for review, [28]). In these theories, for finite graphs usually continuity conditions and Kirchhoff conditions are imposed at interior vertices while Robin conditions at pendant vertices. All these are motivated by physics. Important cases of Robin conditions are Dirichlet and Neumann conditions which we will use in this paper.

To explain our aims let us recall (see, for example, [20, 21]) that the eigenvalues of two self-adjoint Sturm–Liouville boundary value problems

$$-y'' + q(x)y = \lambda^2 y, \tag{1.1}$$

$$y'(0) + Hy(0) = 0, \tag{1.2}$$

$$y'(1) + h_j y(1) = 0, \qquad j = 1, 2, \tag{1.3}$$

with real $q \in L_2(0,l)$, $H \in \mathbf{R} \cup \{\infty\}$, $h_j \in \mathbf{R} \cup \infty$ and $h_1 < h_2$ interlace in strict meaning:

$$-\infty < \left(\lambda_1^{(1)}\right)^2 < \left(\lambda_1^{(2)}\right)^2 < \left(\lambda_2^{(1)}\right)^2 < \left(\lambda_2^{(2)}\right)^2 < \cdots.$$

If $h = 0$, the corresponding boundary condition is said to be Neumann and if $h = \infty$, then it is said to be Dirichlet. This is a particular case of general results obtained in [11] for boundary value problems generated by the general equation of small transversal vibrations of a string.

To solve inverse problems of recovering the potentials on the edges using the spectra of boundary problems it is necessary to characterize spectral data. In the classical case (see [19]), the interlacing property of eigenvalues and certain asymptotics of the spectra give the necessary and sufficient conditions for two sequences to be the spectra of problems (1.1)–(1.3). So we expect that some form of interlacing behavior should be a part of necessary and sufficient conditions for sequences of real numbers to be the spectra of boundary value problems on an arbitrary connected finite metric graph.

In [24], the following analog of classical interlacing was obtained for the inverse three spectral problem. Let $\{\lambda_k\}_{\substack{k\neq 0\\-\infty}}^{\infty}$ be the spectrum of problem:

$$y'' + \lambda^2 y - q(x)y = 0,$$
$$y(0) = y(a) = 0.$$

Here the potential $q(x) \in W_2^1(0,a)$ is real and such that the spectrum is real and does not contain 0.

Let $\{\nu_k\}_{\substack{k\neq 0\\-\infty}}^{\infty}$, $\left\{\nu_k^{(1)}\right\}_{\substack{k\neq 0\\-\infty}}^{\infty}$ be the spectra of problems

$$y'' + \lambda^2 y - q(x)y = 0,$$
$$y(0) = y\left(\frac{a}{2}\right) = 0,$$

and

$$y'' + \lambda^2 y - q(x)y = 0,$$
$$y\left(\frac{a}{2}\right) = y(a) = 0,$$

respectively.

Theorem 1.1.

(a) All λ_k, ν_k and $\nu_k^{(1)}$ are simple, $0 < \lambda_1 < \nu_1$; $\lambda_1 < \nu_1^{(1)}$.

(b) For every $n > 1$ the following alternative is valid: either the interval (λ_1, λ_n) contains exactly $n-1$ (counting multiplicities) elements of the set $\{\nu_k\}_1^\infty \cup \left\{\nu_k^{(1)}\right\}_1^\infty$, and then $\lambda_n \notin \{\nu_k\}_1^\infty \cup \left\{\nu_k^{(1)}\right\}_1^\infty$, or the interval (λ_1, λ_n) contains exactly $n-2$ (counting multiplicities) elements of the set $\{\nu_k\}_1^\infty \cup \left\{\nu_k^{(1)}\right\}_1^\infty$, and then $\lambda_n \in \{\nu_k\}_1^\infty \cap \left\{\nu_k^{(1)}\right\}_1^\infty$.

The proof of this theorem can be found in [25 theorem 1.13]. Generalizations of this theorem to the case of nonequal subintervals and wider class of q's, and boundary conditions were obtained in [9]. For other generalizations and discrete analogues, see [1, 2, 22].

The above spectral problems on the interval $[0, a]$ can be considered as problems on a star graph having two edges of the length $a/2$ each. In this case, we call

$$y\left(\frac{a}{2}-0\right) = y\left(\frac{a}{2}+0\right) = 0$$

the Dirichlet conditions, and

$$y\left(\frac{a}{2}-0\right) = y\left(\frac{a}{2}+0\right),$$
$$y'\left(\frac{a}{2}-0\right) = y'\left(\frac{a}{2}+0\right)$$

the Neumann conditions. Then $\{\lambda_k\}_{k\neq 0, -\infty}^{\infty}$ can be called the Neumann spectrum and $\{\nu_k\}_{k\neq 0, -\infty}^{\infty} \cup \left\{\nu_k^{(1)}\right\}_{k\neq 0, -\infty}^{\infty}$ the Dirichlet spectrum. In this paper, we use generalizations of these notions for trees.

Generalizations of the above interlacing results for star graphs of more than two edges were obtained in [25, 26] for Sturm–Liouville problems, for Stieltjes strings in [3] and for the problem generated by the string equation in [27]. For the case of a graph with simple eigenvalues these results were generalized in [29].

In this paper, we consider the case of trees but allow eigenvalues to be of arbitrary geometric multiplicity. We establish some useful formulae,

$$\phi_N(\lambda^2) = \phi_N^{(1)}(\lambda^2)\phi_D^{(2)}(\lambda^2) + \phi_N^{(2)}(\lambda^2)\phi_D^{(1)}(\lambda^2),$$
$$\phi_D(\lambda^2) = \phi_D^{(1)}(\lambda^2)\phi_D^{(2)}(\lambda^2),$$

connecting characteristic functions of Neumann and Dirichlet problems on whole graph with these functions on subgraphs. These formulae were obtained in [17] when one of the subgraphs is just an edge. Furthermore, the function $\frac{\phi_D}{\phi_N}$ will be analyzed in more detail.

We also study the interlacing behavior of eigenvalues of different boundary value problems on a tree. It is interesting to note that these results are true also in the case of boundary value problems generated by recurrence relations connected with small vibrations of Stieltjes strings.

2. Characteristic functions

Let T be a metric tree with n edges. We denote by v_j the vertices, by $d(v_j)$ their degrees, by e_j the edges and by l_j their lengths. An arbitrary vertex $\mathbf{v}$ is chosen as the root. Local coordinates for edges identify the edge e_j with the interval $[0, l_j]$ so that the local coordinate increases as the distance to the root decreases. This means that each pendant vertex has the local coordinate 0 if it is not the root. The root has local coordinate l_j for any jth edge incident to it. All the other interior vertices v have one outgoing edge, with local coordinate 0, while the local coordinate for v for an incoming edge e_j is l_j. Functions y_j on the edges are subject to a system of n scalar Sturm–Liouville equations:

$$-y_j'' + q_j(x)y_j = \lambda^2 y_j, \tag{2.1}$$

where q_j is a real-valued function which belongs to $L^2[0, l_j]$. For an edge e_j incident to a pendant vertex we impose self-adjoint boundary conditions,

$$y_j'(0) + \beta_j y_j(0) = 0, \tag{2.2}$$

where $\beta_j \in \mathbf{R} \cup \{\infty\}$. The case $\beta_j = \infty$ corresponds to the Dirichlet boundary condition $y_j(0) = 0$.

At the root **v**, we impose the continuity conditions

$$y_j(l_j) = y_k(l_k) \tag{2.3}$$

for all incident edges e_j and e_k, and the Kirchhoff condition

$$\sum_j y_j'(l_j) = 0, \tag{2.4}$$

where the sum is taken over all edges e_j incident to **v**. For all other interior vertices with incoming edges e_j and outgoing edge e_k the continuity conditions are

$$y_j(l_j) = y_k(0), \tag{2.5}$$

and the Kirchhoff condition is

$$y_k'(0) = \sum_j y_j'(l_j). \tag{2.6}$$

Let us denote by $s_j(\lambda, x)$ the solution of the Sturm–Liouville equation (2.1) on the edge e_j which satisfies the conditions $s_j(\lambda, 0) = s_j'(\lambda, 0) - 1 = 0$ and by $c_j(\lambda, x)$ the solution which satisfies the conditions $c_j(\lambda, 0) - 1 = c_j'(\lambda, 0) = 0$. Then the *characteristic function*, i.e. an entire function whose zeros coincide with the spectrum of the problem can be expressed by $s_j(\lambda, l_j), s_j'(\lambda, l_j), c_j(\lambda, l_j)$ and $c_j'(\lambda, l_j)$. To do it we introduce the following system of vector functions $\psi_j(\lambda, x) = \mathrm{col}\{0, 0, \ldots, s_j(\lambda, x), \ldots, 0\}$ and $\psi_{j+n}(\lambda, x) = \mathrm{col}\{0, 0, \ldots, c_j(\lambda, x), \ldots, 0\}$ for $j = 1, 2, \ldots, n$, where n is the number of edges. As in [29], we denote by L_j $(j = 1, 2, \ldots, 2n)$ the linear functionals generated by (2.2)–(2.6). Then $\Phi(\lambda^2) = \|L_j(\psi_k(\lambda, x)\|_{j,k}^{2n}$ is the characteristic matrix which represents the system of linear equations describing the continuity and Kirchhoff conditions for the internal vertices. Then we call

$$\phi_N(\lambda^2) := \det(\Phi(\lambda^2))$$

the *characteristic function* of problem (2.1)–(2.6). The characteristic function is determined up to a constant multiple. For the sake of convenience, we use the spectral parameter $z = \lambda^2$. It is easy to see that the characteristic function satisfies

$$\phi_N(\overline{z}) = \overline{\phi_N(z)}.$$

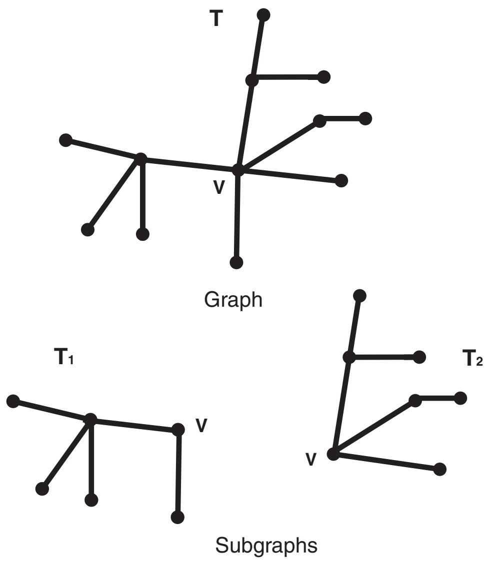
We are interested also in the problem generated by the same equations and the same boundary and matching conditions, but with the condition

$$y_j(l_j) = 0 \tag{2.4$'$}$$

instead of (2.4) at **v**. We denote this characteristic function of problem (2.1)–(2.3), (2.4′), (2.5), (2.6) by $\phi_D(\lambda^2)$. In case when **v** is a pendant vertex, condition (2.4) coincides with the Neumann boundary condition, and condition (2.4′) coincides with the Dirichlet boundary condition. Also $\phi_D(z)$ satisfies the symmetry condition

$$\phi_D(\overline{z}) = \overline{\phi_D(z)}.$$

Let us assume that the root **v** is an interior vertex. We divide our tree T into two subtrees T_1 and T_2 having **v** as the only common vertex. (We say that T_1 and T_2 are *complementary* *subtrees* of T.)



We consider two Neumann problems on the subtrees:

$$-y''_{j,i} + q_{j,i}(x)y_{j,i} = zy_{j,i}, \qquad q_{j,i} \in L^2[0, l_{j,i}], \qquad i = 1, 2. \tag{2.7}$$

For an edge $e_{j,i} \in T_i$ incident to a pendant vertex,

$$y'_{j,i}(0) + \beta_{j,i} y_{j,i}(0) = 0. \tag{2.8}$$

At the root of the subtree T_i, we still have the continuity and Kirchhoff conditions for all incident edges j and k:

$$y_{j,i}(l_{j,i}) = y_{k,i}(0), \tag{2.9}$$

$$\sum_j y'_{j,i}(l_{j,i}) = 0. \tag{2.10}$$

For all other interior vertices with incoming edges $e_{j,i}$ and outgoing edge $e_{k,i}$, the continuity conditions are

$$y_{j,i}(l_{j,i}) = y_{k,i}(0), \tag{2.11}$$

and the Kirchhoff condition is

$$y'_{k,i}(0) = \sum_{j,i} y'_{j,i}(l_{j,i}). \tag{2.12}$$

The two Dirichlet problems on T_1 and T_2 are

$$-y''_{j,i} + q_{j,i}(x)y_{j,i} = \lambda y_{j,i}, \qquad q_{j,i} \in L^2[0, l_{j,i}], \qquad i = 1, 2. \tag{2.13}$$

For an edge $e_{j,i}$ incident to a pendant vertex, we let

$$y'_{j,i}(0) + \beta_{j,i} y_{j,i}(0) = 0. \tag{2.14}$$

At the root of T_i,

$$y_{j,i}(l_{j,i}) = 0, \tag{2.15}$$

for all edges $e_{j,i}$ belonging to the part i incident to the root. For all other interior vertices with incoming edges $e_{j,i}$ and outgoing edge $e_{k,i}$ the continuity conditions are

$$y_{j,i}(l_{j,i}) = y_{k,i}(0), \tag{2.16}$$

and the Kirchhoff condition is

$$y'_{k,i}(0) = \sum_{j,i} y'_{j,i}(l_{j,i}). \tag{2.17}$$

Denote by $\phi_N^{(i)}(z)$ the characteristic function of problem (2.7)–(2.12) and by $\phi_D^{(i)}(z)$ the characteristic function of problem (2.13)–(2.17). With these terminologies, we have the following reduction formula for the characteristic functions of a tree.

Theorem 2.1. *Let the root* **v** *of a tree T be an interior vertex. Let T_1 and T_2 be the complementary subtrees of T. Then with the same orientation of the graph and the subgraphs edges described above,*

$$\begin{aligned} \phi_N(z) &= \phi_N^{(1)}(z)\phi_D^{(2)}(z) + \phi_D^{(1)}(z)\phi_N^{(2)}(z), \\ \phi_D(z) &= \phi_D^{(1)}(z)\phi_D^{(2)}(z). \end{aligned} \tag{2.18}$$

Proof. Fix the edges $e_j \in T_1$ and $e_k \in T_2$ incident to the root **v**. Without loss of generality, we let both e_j and e_k be the internal edges. (The case when they are boundary edges is even simpler.) Then the characteristic matrix $\Phi(\lambda^2)$ can be expressed as

$$\Phi(\lambda^2) = \left[\begin{array}{cccc|cccc} * & \cdots & \cdots & * & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & \cdots & * & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & s_j(l_j) & c_j(l_j) & -s_k(l_k) & -c_k(l_k) & \cdots & 0 \\ \hline * & \cdots & s'_j(l_j) & c'_j(l_j) & s'_k(l_k) & c'_k(l_k) & \cdots & * \\ 0 & \cdots & \cdots & 0 & * & \cdots & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & * & \cdots & \cdots & * \end{array}\right].$$

Here the upper left square submatrix describes the continuity and Kirchhoff conditions at the vertices in T_1. So its determinant is $\phi_D^{(1)}(\lambda^2)$, for the last row demonstrates the Dirichlet condition at **v**. The lower right square submatrix describes those conditions in T_2 with the Neumann boundary condition at **v**. So its determinant is $\phi_N^{(2)}(\lambda^2)$. What remain in $\det\Phi(\lambda^2)$ is the product of the determinants of the upper left submatrix and lower right submatrix of the matrix formed by interchanging the middle two row vectors concerning the continuity and Kirchhoff conditions at **v**. Hence the overall characteristic function $\phi_N(z)$ is given by

$$\begin{aligned} \phi_N(\lambda^2) &= \det\Phi(\lambda^2) \\ &= \phi_D^{(1)}(\lambda^2)\phi_N^{(2)}(\lambda^2) - \det\begin{bmatrix} * & \cdots & \cdots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & \cdots & \cdots & * \\ * & \cdots & s'_j(l_j) & c'_j(l_j) \end{bmatrix} \cdot \det\begin{bmatrix} -s_k(l_k) & -c_k(l_k) & \cdots & 0 \\ * & \cdots & \cdots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & \cdots & \cdots & * \end{bmatrix} \\ &= \phi_D^{(1)}(\lambda^2)\phi_N^{(2)}(\lambda^2) + \phi_N^{(1)}(\lambda^2)\phi_D^{(2)}(\lambda^2). \end{aligned}$$

Equation (2.18) is evident from the definition of ϕ_D. □

Corollary 2.2. *Suppose a tree T with root $\mathbf{v}$ has $d(\mathbf{v})$ complementary subtrees T_i $(i = 1, 2, \ldots, d(v))$. Let $\phi_N^{(i)}$ and $\phi_D^{(i)}$ denote the Neumann and the Dirichlet characteristic functions for T_i. If $\mathbf{v}$ is a pendant vertex for T_i, then*

$$\phi_N(z) = \sum_{i=1}^{d(\mathbf{v})} \phi_N^{(i)}(z) \prod_{i=1,k\neq i}^{d(\mathbf{v})} \phi_D^{(k)}(z), \tag{2.19}$$

$$\phi_D(z) = \prod_{i=1}^{d(v)} \phi_D^{(i)}(z). \tag{2.20}$$

Another application of theorem 2.1 lies in the understanding of the complex function $\frac{\phi_D}{\phi_N}$. For this, we need the notion of a Nevanlinna function. It is also called the R-function [11] or Herglotz function, and its definition also varies. Below is the definition we use in this paper.

Definition. *A meromorphic function $f(z)$ is said to be a Nevanlinna function if:*

(i) it is analytic in the half-planes $\operatorname{Im} z > 0$ *and* $\operatorname{Im} z < 0$,
(ii) $f(\overline{z}) = \overline{f(z)}$ $(\operatorname{Im} z \neq 0)$,
(iii) $\operatorname{Im} z \operatorname{Im} f(z) \geqslant 0$ *for* $\operatorname{Im} z \neq 0$.

The following lemma is obvious.

Lemma 2.3. *Suppose that f and g are Nevanlinna functions, then $f + g$ and $-\frac{1}{f}$ are also Nevanlinna functions.*

Theorem 2.4. *The ratio*

$$\frac{\phi_D(z)}{\phi_N(z)}$$

is a Nevanlinna function.

Proof. We extend to the case of the graph domain the well-known method (see [10]). Using Lagrange identity (see [23 part II, p 50]) for solution y_j we obtain

$$-\mathrm{i}(-y_j'\overline{y_j} + \overline{y_j}'y_j)|_0^{l_j} = 2\operatorname{Im} z \int_0^{l_j} |y_j|^2\,\mathrm{d}x. \tag{2.21}$$

If T is just an interval, then using (2.2) we obtain from (2.21)

$$-\mathrm{i}(-y'(l_j)\overline{y_j(l_j)} + \overline{y_j'(l_j)}y_j(l_j)) = 2\operatorname{Im} z \int_0^{l_j} |y_j|^2\,\mathrm{d}x.$$

Thus on an interval,

$$-\operatorname{Im}\left(\frac{\phi_N(z)}{\phi_D(z)}\right) = -\operatorname{Im}\left(\frac{y_j'(l_j)}{y_j(l_j)}\right) = \operatorname{Im} z \frac{\int_0^{l_j} |y_j|^2\,\mathrm{d}x}{|y_j(l_j)|^2}.$$

It means that $\frac{\phi_D(z)}{\phi_N(z)}$ in the case of an interval is a Nevanlinna function.

For a general tree, we can use the root $\mathbf{v}$ to break down the tree into complementary subtrees T_i's $(i = 1, \ldots, d(\mathbf{v}))$. By theorem 2.1,

$$\frac{\phi_N(z)}{\phi_D(z)} = \sum_{i=1}^{d(\mathbf{v})} \frac{\phi_N^{(i)}(z)}{\phi_D^{(i)}(z)}.$$

Then

$$\frac{\phi_D(z)}{\phi_N(z)} = -\left(-\sum_{i=1}^{d(\mathbf{v})} \frac{\phi_N^{(i)}(z)}{\phi_D^{(i)}(z)}\right)^{-1}.$$

By lemma 2.3, $\frac{\phi_D}{\phi_N}$ is a Nevanlinna function whenever $\frac{\phi_D^{(1)}}{\phi_N^{(1)}}$ and $\frac{\phi_D^{(2)}}{\phi_N^{(2)}}$ are both Nevanlinna functions. Using a recursive argument, the statement is valid for any tree. □

Definition (see [11]). *A Nevanlinna function $f(z)$ is said to be an S-function if $f(z) > 0$ for $z < 0$.*

Theorem 2.5. *There exists a number $\beta > 0$ such that*

$$\frac{\phi_D(z-\beta)}{\phi_N(z-\beta)}$$

is an S-function.

Proof. In view of theorem 2.4 it is sufficient to show that there exists $\beta > 0$ such that $\frac{\phi_D(z)}{\phi_N(z)} > 0$ for $z < \beta$. Using corollary 2.2, we obtain

$$\frac{\phi_N(z)}{\phi_D(z)} = \sum_{j=1}^{d(v)} \frac{\phi_N^{(j)}(z)}{\phi_D^{(j)}(z)}. \tag{2.22}$$

All the functions $\phi_N^{(j)}(z)$ and $\phi_D^{(j)}(z)$ are the characteristic functions with Neumann and Dirichlet conditions, respectively, at the pendant root. Therefore,

$$\frac{\phi_N^{(j)}(z)}{\phi_D^{(j)}(z)} = \frac{c_j(\sqrt{z},l_j)\tilde{\phi}_N^{(j)}(z) + c_j'(\sqrt{z},l_j)\tilde{\phi}_D(z)}{s_j(\sqrt{z},l_j)\tilde{\phi}_N^{(j)}(z) + s_j'(\sqrt{z},l_j)\tilde{\phi}_D(z)}. \tag{2.23}$$

The asymptotic expansions for $s_j(\sqrt{z},l_j)$, $s_j'(\sqrt{z},l_j)$, $c_j(\sqrt{z},l_j)$, $c_j'(\sqrt{z},l_j)$ are well known (see, for example, [21]):

$$s_j(\sqrt{z},l_j) \underset{z\to-\infty}{=} \frac{\mathrm{e}^{\sqrt{|z|}l_j}}{2\sqrt{|z|}}(1+o(1)), \qquad s_j'(\sqrt{z},l_j) \underset{z\to-\infty}{=} \frac{\mathrm{e}^{\sqrt{|z|}l_j}}{2}(1+o(1)),$$

$$c_j(\sqrt{z},l_j) \underset{z\to-\infty}{=} \frac{\mathrm{e}^{\sqrt{|z|}l_j}}{2}(1+o(1)), \qquad c_j'(\sqrt{z},l_j) \underset{z\to-\infty}{=} \frac{\sqrt{|z|}\,\mathrm{e}^{\sqrt{|z|}l_j}}{2}(1+o(1)).$$

Using these asymptotics we obtain from (2.23)

$$\frac{\phi_N^{(j)}(z)}{\phi_D^{(j)}(z)} \underset{z\to-\infty}{=} \sqrt{|z|}(1+o(1)).$$

Therefore, all the summands in (2.22) are positive for negative z with $|z|$ large enough. Thus, there exists $\beta > 0$ such that $z < -\beta$ implies $\frac{\phi_N(z)}{\phi_D(z)} > 0$ and $\frac{\phi_D(z)}{\phi_N(z)} > 0$. □

Denote by $\{\zeta_k\} = \{\nu_k^{(1)}\} \cup \{\nu_k^{(2)}\}$ and by $\{\xi_k\} = \{\mu_k^{(1)}\} \cup \{\mu_k^{(2)}\}$, where $\{\nu_k^{(i)}\}$ is the set of zeros of $\phi_D^{(i)}(\lambda^2)$, and $\{\mu_k^{(i)}\}$ is the set of zeros of $\phi_N^{(i)}(\lambda^2)$.

Corollary 2.6. *The zeros λ_k of $\phi_N(\lambda^2)$ interlace with $\{\zeta_k\}$ and with $\{\xi_k\}$ as follows:*

$$\lambda_1^2 \leqslant \zeta_1^2 \leqslant \lambda_2^2 \leqslant \zeta_2^2 \leqslant \cdots$$
$$\xi_1^2 \leqslant \lambda_1^2 \leqslant \xi_2^2 \leqslant \lambda_2^2 \leqslant \cdots.$$

Proof. By theorem 2.1

$$-\frac{\phi_N(z)}{\phi_D^{(1)}(z)\phi_D^{(2)}(z)} = -\frac{\phi_N^{(1)}(z)}{\phi_D^{(1)}(z)} - \frac{\phi_N^{(2)}(z)}{\phi_D^{(2)}(z)} \tag{2.24}$$

and

$$\frac{\phi_N(z)}{\phi_N^{(1)}(z)\phi_N^{(2)}(z)} = \frac{\phi_D^{(1)}(z)}{\phi_N^{(1)}(z)} + \frac{\phi_D^{(2)}(z)}{\phi_N^{(2)}(z)}. \tag{2.25}$$

By theorem 2.4 both terms at the right-hand side of (2.24) and (2.25) are Nevanlinna functions and so are the associated left-hand sides. The statement thus follows from the interlacing properties of Nevanlinna functions [18] and theorem 2.5. □

These interlacing results are also evident from considering such boundary value problems either variationally or via matrix Prüffer angles (see [8, 6]); both of the approaches are different from that used in this paper.

3. Multiplicities of eigenvalues

Theorem 3.1. *Denote by $p_N(z)$ the multiplicity of z as a zero of ϕ_N and by $p_D(z)$ the multiplicity of z as a zero of $\phi_D(z)$. Then*

(a) $|p_N(z) - p_D(z)| \leqslant 1$,
(b) $p_N(z) \leqslant n - n_i$, *where n is the number of edges and n_i is the number of interior vertices.*

Proof. Part (a) follows from corollary 2.6. For part (b), we first observe that the multiplicity of any nonzero eigenvalue $z = \lambda^2$ is equal to $d_N(\lambda) := \dim(\mathrm{Ker}(\Phi(\lambda)))$, the dimension of the nullspace of the characteristic matrix $\Phi(\lambda)$, for if any zero of an entry in $\Phi(\lambda)$, if exists, is simple. We shall use induction on n. The case $n = 1$ is obvious. When $n = 2$, then $n_i = 1$, and

$$\Phi_2(\lambda) = \begin{pmatrix} S_1(\lambda) & -S_2(\lambda) \\ S_1'(\lambda) & S_2'(\lambda) \end{pmatrix}. \tag{3.1}$$

So if λ is a zero of the row vector $(S_1(\lambda), -S_2(\lambda))$, then it would not be a zero of the other row vector. Thus $d_N \leqslant 1$. Suppose the statement is true for $n = k$ with each $n_i \leqslant k - 1$. Then any tree with $k + 1$ edges (and n_i interior vertices) can be obtained by adding one more edge to a k-edged tree. If the $(k + 1)$th edge is added to a pendent vertex, the graph can be considered as unchanged. Thus by induction hypothesis,

$$d_N \leqslant k - (n_i - 1) = (k + 1) - n_i.$$

Suppose that the $(k+1)$th edge is added to an interior vertex, the dimension of the characteristic matrix increases by 1, with an additional row such as $(S_1, -S_2, 0, \ldots, 0)$ and an additional column. Hence,

$$d_N \leqslant \dim(\mathrm{Ker}(\Phi_k(\lambda))) + 1 \leqslant k + 1 - n_i.$$

The statement is valid for any $n \in \mathbf{N}$ by mathematical induction. □

Corollary 3.2.

(a) If the root **v** *is an interior vertex then* $p_D(z) \leqslant n - n_i + 1$.

(b) *If* **v** *is a pendant vertex then*

$$p_D(z) + p_N(z) \leqslant 2n - 2n_i - 1.$$

Proof. Part (a) follows from theorem 3.1. For part (b), we may assume that T has at least one vertex with degree greater than 2. For if not, T is an interval, where the Dirichlet and Neumann eigenvalues do not overlap, and $p_D(z) + p_N(z) \leqslant 1$.

So let the pendant vertex be incident to the edge e_1, and v_{k+1} be the first vertex degree greater than 2, after passing through edges $\{e_i : 1 \leqslant i \leqslant k\}$. Now suppose $p_N(z) \geqslant n - n_i$, $p_D(z) \geqslant n - n_i$ for some z. By (2.19), we obtain that z is a zero of multiplicity at least $n - n_i$ of

$$\phi_N(z) = c_1(\sqrt{z}, l_1 + l_2 + \cdots + l_k)\phi_N^{(2)}(z) + c_1'\big(\sqrt{z}, l_1 + l_2 + \cdots + l_k\big)\phi_D^{(2)}(z)$$

and of

$$\phi_D(z) = s_1(\sqrt{z}, l_1 + l_2 + \cdots + l_k)\phi_N^{(2)}(z) + s_1'\big(\sqrt{z}, l_1 + l_2 + \cdots + l_k\big)\phi_D^{(2)}(z).$$

The Lagrange identity gives

$$c_1(\sqrt{z}, l_1 + l_2 + \cdots + l_k)s_1'(\sqrt{z}, l_1 + l_2 + \cdots + l_k) \\ - c_1'(\sqrt{z}, l_1 + l_2 + \cdots + l_k)s_1(\sqrt{z}, l_1 + l_2 + \cdots + l_k) = 1.$$

Therefore, if z is a zero of multiplicity not less than $n - n_i$ for both ϕ_N and ϕ_D, then it is a zero of the same multiplicity of $\phi_N^{(2)}$ and $\phi_D^{(2)}$ which is impossible since the tree T_1 obtained from T by deleting k edges connecting the root **v** with v_{k+1} has $n - k$ edges and $n_i - k + 1$ interior vertices, and according to (b) of theorem 3.1 the multiplicity $p_N^{(2)}$ of the Neumann problem for T_1 satisfies inequality $p_N^{(2)} \leqslant n - k - (n_i - k + 1) = n - n_i - 1$. Thus (b) is proved. □

Corollary 3.3. *Let the root be an interior vertex. Then*

(a) *The multiplicity of any nonzero eigenvalue of problem (2.2)–(2.6) does not exceed* $n - n_i$.
(b) *The multiplicity of any nonzero eigenvalue of problem (2.2), (2.3), (2.4′), (2.5), (2.6) does not exceed* $n - n_i + 1$.
(c) *If* 0 *is an eigenvalue of problem (2.2)–(2.6) then its geometric multiplicity does not exceed* $n - n_i$ *and its algebraic multiplicity is twice more.*
(d) *If* 0 *is an eigenvalue of problem (2.2), (2.3), (2.4′), (2.5), (2.6) then its geometric multiplicity does not exceed* $n - n_i + 1$ *and its algebraic multiplicity is twice more.*

Theorem 3.4. *Suppose a tree T rooted at* **v** *has complementary subtrees T_j $(j = 1, 2, \ldots, d)$, and $\bigcup_{j=1}^{d} V_{T_j} = V_T$ and $V_{T_j} \cap V_{T_r} = \{\mathbf{v}\}$. Let $\phi_N(z)$ and $\phi_D(z)$ be as in (2.19) and (2.20). Suppose that z_0 is a common zero of $\phi_N(z)$ and $\phi_D(z)$ with multiplicities p_N and p_D, respectively. Then $p_N \geqslant p_D$ implies that $p_D \leqslant n - d - n_i + 1$.*

Proof. According to corollary 3.2 for each j

$$p_D^{(j)}(z) + p_N^{(j)}(z) \leqslant 2n^{(j)} - 2n_i^{(j)} - 1, \tag{3.2}$$

where $n^{(j)}$ is the number of the edges of the jth subtree, $n_i^{(j)}$ is the number of the interior vertices of the jth subtree. According to (2.22), $p_N(z) = \min\limits_{1\leqslant j\leqslant d} \big\{p_N^{(j)}(z) + \sum_{k=1,k\neq j}^{d} p_D^{(k)}(z)\big\}$ and $p_D(z) = \sum_{k=1}^{d} p_D^{(k)}(z)$, it is clear that the condition $p_N(z) \geqslant p_D(z)$ implies

$$p_N^{(j)}(z) \geqslant p_D^{(j)}(z) \tag{3.3}$$

for each j. Combining (3.2) with (3.3) we obtain

$$p_D^{(j)}(z) \leqslant n^{(j)} - n_i^{(j)} - 1.$$

Using this inequality we get

$$p_D(z) = \sum_{j=1}^{d} p_D^{(j)}(z) \leqslant \sum_{j=1}^{d} n^{(j)} - \sum_{j=1}^{d} n_i^{(j)} - d = n - n_i - d + 1. \qquad \square$$

Let us consider a star-shaped graph of n edges rooted at the interior vertex. Then

$$\phi_N(z) = \sum_{j=1}^{n} (c_j'(\sqrt{z}, l_j) - \beta_j s_j'(\sqrt{z}, l_j)) \prod_{k=1, k\neq j}^{n} (c_k(\sqrt{z}, l_k) - \beta_k s_k(\sqrt{z}, l_k)),$$

$$\phi_D(z) = \prod_{k=1}^{n} (c_k(\sqrt{z}, l_k) - \beta_k s_k(\sqrt{z}, l_k)).$$

Denote by $\{\lambda_k^2\}$ the set of zeros of $\phi_N(z)$ and by $\{\theta_k^2\} = \bigcup_{j=1}^{n} \{\tau_k^{(j)2}\}$ the union of sets $\{\tau_k^{(j)2}\}$ of zeros of $c_j(\sqrt{z}, l_j) - \beta_j s_j(\sqrt{z}, l_j)$.

Corollary 3.5.

(a) $\lambda_1^2 \leqslant \theta_1^2 \leqslant \lambda_2^2 \leqslant \theta_2^2 \leqslant \cdots$.
(b) $\theta_k^2 = \lambda_{k+1}^2$ *if and only if* $\lambda_{k+1}^2 = \theta_{k+1}^2$.
(c) Multiplicity of θ_k^2 *does not exceed* n.

Proof. Part (a) follows from corollary 2.6. Part (c) follows from simplicity of zeros of $c_j(\sqrt{z}, l_j) - \beta_j s_j(\sqrt{z}, l_j)$. To prove (b), we note that for a star graph $n_i = 1$ and $d = n$. Therefore, theorem 3.4 implies that if z is a common zero for ϕ_D and ϕ_N then $p_N < p_D$. Then it follows from theorem 3.1 that $p_N = p_D - 1$. This together with the interlacing property implies (b). $\square$

Thus we have obtained theorem 3.2 in [26] as a particular case.

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